Gain and saturation in free-electron laser oscillators

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We present a calculation of the free-electron laser (FEL) gain, in the small gain, large signal regime that is of relevance to FEL oscillators. We derive an analytic expression for the gain as a function of both the detuning parameter and the optical intensity. We use this analysis to analytically determine the parameters of a three-parameter generalized gain function that is valid to larger intensities and into the saturation regime. We find that with increasing optical intensity the peak of the detuning curve shifts towards higher values of the detuning parameter, and that at higher detuning the gain can actually increase with intensity before falling. We use this parametrization to predict the dependence of the saturated power in an FEL oscillator on the detuning parameter. Our results are in good agreement with one-dimensional numerical simulations that do *not* make the small gain approximation and also with experimental data from the Institute of High Energy Physics, Beijing and Stanford University FIREFLY FELs. Our analysis should therefore be useful in the design of FEL oscillators. [S1063-651X(97)10802-9]

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I. INTRODUCTION

The free-electron laser (FEL) [1] produces widely tunable coherent radiation by passing a beam of electrons from an accelerator through a static, spatially periodic, magnetic field, produced by a device called a wiggler. The wavelength λ_R of the radiation is given by

$$\lambda_R = \frac{\lambda_W}{2\gamma_R^2}(1+a_W^2), \tag{1}$$

where λ_W is the wavelength of the periodic magnetic field, γ_R is the resonant energy of the electron in units of its rest mass m_e, and a_W $(=e\lambda_W B_W/2\pi m_e c)$ is the *wiggler parameter*. Here B_W is the rms value of the magnetic field, e is the charge of the electron, and c is the speed of light in vacuum.

Actually, electrons with energy exactly equal to γ_R are in resonance with the optical field and do not, on average, exchange energy with it. In order to have gain in the system it is necessary to ''detune'' the energy, and the deviation from resonance is measured by the detuning $\mu = 4\pi N_W(\gamma-\gamma_R)/\gamma_R$, which is just the electron energy measured relative to the resonant value and normalized to the energy bandwidth of the FEL; here N_W is the number of wiggler periods. The value of μ at the entrance to the wiggler is called the *detuning parameter* α. Clearly, α is an important parameter in determining the gain, and hence the performance, of the FEL.

In the limit that the gain and the optical intensity are small, it is a textbook calculation to determine the *small-signal-gain (SSG) curve* $g_0(\alpha)$ as a function of the detuning parameter [1]. This *detuning curve* is asymmetric and has a maximum at $\alpha=2.6$. One of the early successes of FEL theory came when the detuning curve was measured and found to agree well with theory [2]. The SSG analysis is very helpful in understanding the startup of an FEL oscillator. For the FEL to lase, the SSG is required to be greater than the total round-trip loss. Once the FEL starts lasing, the intracavity power in the FEL starts building up. It is well known that the gain of an FEL, away from the SSG limit, depends on the intracavity intensity. As the intensity builds up the gain falls. Finally, when the gain becomes equal to the loss, the optical power saturates. It is important to realize that away from the SSG limit the functional dependence of the gain on the detuning parameter changes; in particular the peak of the detuning curve will no longer be at $\alpha=2.6$. Hence, to better understand the growth and saturation of optical power in an FEL, it is necessary to derive the functional relationship of the gain on both detuning and intensity, i.e., to determine a *generalized gain function* that in the SSG limit reduces to the well-known detuning curve.

FELs can be operated in two different configurations: as amplifiers and as oscillators. The former are single-pass devices and, for that reason, operate with high gain. They also require long undulators, and are therefore expensive devices. FEL oscillators are multi-pass devices. The gain per pass may be low, but many electron bunches pass through the undulator and the optical power can therefore, over many passes, build up to a large value. Most operating FELs are oscillators.

There have been earlier analytic studies of FEL dynamics and gain in the high gain regime that are of relevance to FEL amplifiers [3,4]. In these analyses the emphasis is on the influence of three-dimensional effects (such as transverse emittance and betatron motion) on the gain of FEL amplifiers. Their approach is to treat the electron beam as continuous and derive a dispersion relation from the Maxwell-Vlasov equations. In Ref. [3] the dispersion relation is solved using a variational technique, whereas in Ref. [4] it is solved by expansion in a complete set of orthogonal functions. The final solution, in both cases, is numerical, and hence the functional dependence of the gain on detuning and intensity is difficult to reconstruct. Additionally, in Ref. [4] it is clearly stated that *their numerical solutions do not agree with multiparticle simulations in the one-dimensional limit.* This suggests that the work of Refs. [3,4], though powerful and useful in the design of FEL amplifiers when three-dimensional effects dominate, has limited use in studying purely one-dimensional effects and in the study of FEL oscillators.

In FEL oscillators, where the gain per pass is typically low but the optical power builds up to a large value over many passes, it is the *small gain, large signal* regime that is important. In this paper we consider FEL dynamics and gain in the one-dimensional, small gain, but large signal regime. Our emphasis is on understanding the simultaneous dependence of the gain on the detuning parameter as well as the intensity of the optical field. We stick to the simple, single-particle approach of Colson [5], but extend it to the large signal regime. Our approach provides a transparent and physically intuitive understanding of the FEL gain in the small gain, large signal regime. In the next section we start with the Colson equations [5] and derive an analytic expression for the gain as a function of energy detuning as well as intensity. In Sec. III we use this expression to analytically fix the parameters of a three-parameter parametrization of the gain, the *generalized gain function*, that has a broader regime of validity. In Sec. IV we employ our analytic results to study the buildup of intensity in an FEL oscillator, and use the generalized gain function to predict the saturated intensity in the oscillator. In Sec. V we compare our predictions with data from two operating FEL oscillators, the Beijing FEL and the FIREFLY FEL at Stanford, and show that our predictions are in good agreement with experiments. We conclude with some comments and discussion.

II. DERIVATION OF THE GAIN IN THE SMALL GAIN, LARGE SIGNAL REGIME

We start with the dimensionless Colson equations [5]

$$\frac{d\mu}{d\tau} = -\epsilon_L \sin\psi, \tag{2a}$$

$$\frac{d\psi}{d\tau} = \mu, \tag{2b}$$

$$\frac{d\epsilon_L}{d\tau} = j_e \langle \sin\psi \rangle, \tag{2c}$$

where ψ is the electron's phase relative to the electromagnetic field, ϵ_L is the dimensionless laser field, τ is the dimensionless time, j_e is the dimensionless current density, and $\langle\,\rangle$ indicates averaging over all electrons. Solutions of Eqs. (2) give a good description of the physics of short wavelength FELs, in the one-dimensional limit. For N electrons there are $2N+1$ coupled nonlinear differential equations, which cannot be solved analytically. Analytic solutions are possible only under certain approximations. One approximation that is relevant to FEL oscillators is the *small gain* approximation, where we assume that the gain in a single pass is small. Then ϵ_L in Eqs. (2a) and (2b) can be treated as a constant during any given pass, and these equations consequently decouple from Eq. (2c). Note that the magnitude of ϵ_L can still be large, so that these equations are valid in the *large signal, small gain* limit.

From Eqs. (2a) and (2b) the differential power gain of the FEL can be written as

$$\frac{dG}{d\tau} = \frac{2j_e}{\epsilon_L} \langle \sin\psi \rangle. \tag{3}$$

We look for a solution to Eqs. (2a) and (2b) in the form of the following power series in the electric field ϵ_L:

$$\mu = \mu_0 + \epsilon_L \mu_1 + \epsilon_L^2 \mu_2 + \epsilon_L^3 \mu_3 + \epsilon_L^4 \mu_4 + \epsilon_L^5 \mu_5 + \cdots, \tag{4a}$$

$$\psi = \psi_0 + \epsilon_L \psi_1 + \epsilon_L^2 \psi_2 + \epsilon_L^3 \psi_3 + \epsilon_L^4 \psi_4 + \epsilon_L^5 \psi_5 + \cdots. \tag{4b}$$

One can similarly expand the expression for the gain in powers of ϵ_L; up to fourth order in ϵ_L we get

$$\frac{dG}{d\tau} = \frac{dG_0}{d\tau} + \epsilon_L \frac{dG_1}{d\tau} + \epsilon_L^2 \frac{dG_2}{d\tau} + \epsilon_L^3 \frac{dG_3}{d\tau} + \epsilon_L^4 \frac{dG_4}{d\tau}. \tag{5}$$

Note that the usual SSG analysis only keeps terms up to first order in ϵ_L in Eqs. (4), and hence only the first, intensity-independent, term in Eq. (5). We keep terms up to ϵ_L^4 in order to make the gain a nonlinear function of the intensity [see Eq. (7) below].

Substituting Eqs. (4) back into Eqs. (2a) and (2b), expanding the trigonometric term, and gathering equal powers of ϵ_L on both sides, gives a hierarchy of equations for the various ψ_n and μ_n (n = 0–5). These are solved using the initial conditions $\mu_0 = \alpha$, $\psi_0 = \phi_0$, $\mu_{n>0} = 0$, $\psi_{n>0} = 0$. Here α is the detuning parameter, and ϕ_0 is the electron's initial random phase. We have assumed an ideal monoenergetic unbunched beam.

Similarly, substituting Eqs. (4b) and (5) in Eq. (3) and equating equal powers of ϵ_L gives a set of equations for the various $dG_n/d\tau$; these will be zero for odd n, since the gain depends only on the intensity, i.e., on even powers of ϵ_L. The final expressions are

$$\frac{dG_0}{d\tau} = 2j_e \langle \psi_1 \cos(\alpha\tau + \phi_0) \rangle, \tag{6a}$$

$$\frac{dG_2}{d\tau} = 2j_e \langle (\psi_3 - \psi_1^3/6)\cos(\alpha\tau + \phi_0) - \psi_2\psi_1 \sin(\alpha\tau + \phi_0) \rangle, \tag{6b}$$

$$\frac{dG_4}{d\tau} = 2j_e \left\langle \left(\psi_5 - \frac{\psi_3\psi_1^2}{2} - \frac{\psi_2^2\psi_1}{2} + \frac{\psi_1^5}{120} \right)\cos(\alpha\tau + \phi_0) + \left(\frac{\psi_2\psi_1^3}{6} - \psi_3\psi_2 - \psi_4\psi_1 \right)\sin(\alpha\tau + \phi_0) \right\rangle, \tag{6c}$$

where the various ψ_n are now known quantities.

The total FEL gain (defined here as the ratio of the increase in intensity to the initial intensity) can be evaluated by averaging Eqs. (6) over the initial phase ϕ_0 of the electrons, and then integrating over the entire length of the wiggler (i.e., from τ = 0 to 1). Assuming that the laser field is uniform inside the wiggler during integration (small gain approximation), we find that the FEL gain is given by

$$G(\alpha, \epsilon_L^2) = 2j_e [g_0(\alpha) + \epsilon_L^2 g_2(\alpha) + \epsilon_L^4 g_4(\alpha)], \tag{7}$$

where

$$g_0(\alpha)=\frac{1}{\alpha^3}\left[1-\cos\alpha-\frac{\alpha}{2}\sin\alpha\right], \tag{8a}$$

$$g_2(\alpha)=\frac{1}{8\alpha^7}\left[33-\left(24-\frac{13}{2}\alpha^2\right)\cos\alpha-(9-\alpha^2)\cos2\alpha-\left(\frac{53}{2}\alpha-\frac{1}{2}\alpha^3\right)\sin\alpha-\frac{11}{2}\alpha\sin2\alpha\right], \tag{8b}$$

$$\begin{aligned}g_4(\alpha)=\frac{1}{13824\alpha^{11}}[&(200080+1728\alpha^2)-(76070-58296\alpha^2-288\alpha^4)\cos\alpha-(107152-48960\alpha^2+576\alpha^4)\cos2\alpha\\&-(16858-4176\alpha^2)\cos3\alpha-(154033\alpha-7272\alpha^3+36\alpha^5)\sin\alpha\\&-(122896\alpha-8568\alpha^3)\sin2\alpha-(13845\alpha-468\alpha^3)\sin3\alpha].\end{aligned} \tag{8c}$$

The calculations of the functions $g_2(\alpha)$ and $g_4(\alpha)$, although straightforward, are tedious. The function $g_2(\alpha)$ was calculated manually, but the calculation of the function $g_4(\alpha)$ was performed using MATHEMATICA [6]. The large numbers in the expression for $g_4(\alpha)$ are a consequence of carrying out the calculation to high (fourth) order in ϵ_L.

Equation (7) gives the gain as a function of both energy detuning as well as optical intensity. Note that the gain depends only on the dimensionless parameters α and ϵ_L^2. In principle it also depends on the dimensionless current density j_e; however, that only serves to set the overall scale factor for the gain curve. Thus, Eq. (7) gives a universal family of generalized detuning curves that replace the single SSG curve when the small signal approximation can no longer be made.

Figure 1 shows plots of the functions $g_0(\alpha)$, $g_2(\alpha)$, and $g_4(\alpha)$; note the different scales. At zero detuning they are all zero, resulting in no gain. At very low intensities ($\epsilon_L^2<1$) only g_0 contributes, and the total gain therefore increases to a maximum at $\alpha=2.6$ before decreasing. As ϵ_L^2 increases first g_2 and then g_4 start contributing, and one expects the peak of the gain curve to shift away from $\alpha=2.6$. This is seen explicitly in Fig. 2, which shows the family of detuning curves for different ϵ_L^2. It can be seen that the peak gain decreases with increasing intensity, and also shifts towards $\alpha=4.0$. Figure 2 also shows the corresponding curves obtained from a one-dimensional, time independent, multiparticle FEL simulation that solves the nonlinear Kroll-Morton-Rosenbluth equations [7] and does not make the small gain approximation. The FEL parameters chosen are $\lambda_R=1\,\mu$m, $\lambda_W=4$ cm, $\gamma_R=200$, $a_W=1.0$, $r_{\rm beam}=2$ mm, $N_W=50$, and $I=100$ A. It can be seen that the agreement is very good up to around $\epsilon_L^2=35$, well beyond the SSG regime (for which $\epsilon_L^2<1$).

The shift in the peak of the detuning curve as a function of intensity has an important consequence for FEL oscillators. We know that a free-running oscillator picks a detuning at which the gain is maximum. The operating value of α for an FEL oscillator at saturation is going to be different from 2.6, which is the value of α only during startup. This will introduce a *chirp* in the frequency of the optical radiation. We will show later that it also significantly affects the calculation of maximum saturated power in the FEL oscillator.

III. THE GENERALIZED GAIN FUNCTION $G(\alpha,\epsilon_L^2)$

Figure 2 shows that beyond $\epsilon_L^2=50$ the analytic expression for the gain, Eq. (7), fails. This is because at larger intensities the higher-order terms that we have dropped in the

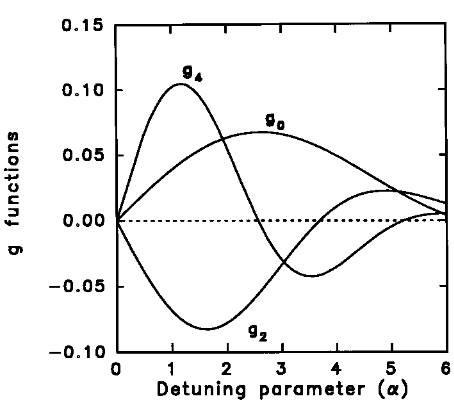

FIG. 1. Plots of $g_0(\alpha)$, $g_2(\alpha)$, and $g_4(\alpha)$ as a function of the detuning parameter α [Eqs. (8)]. For the sake of comparision, $g_2(\alpha)$ and $g_4(\alpha)$ are enhanced by a factor of 10^2 and 10^4, respectively.

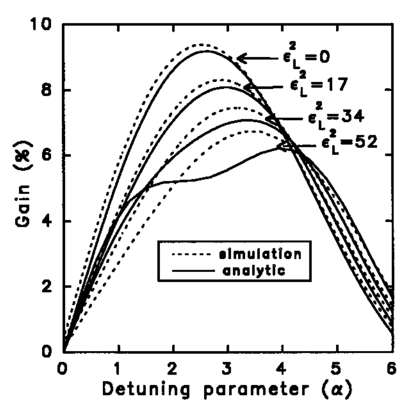

FIG. 2. Plot of the total gain as a function of the detuning parameter α for different intensities ϵ_L^2 for the analytic calculation [Eq. (7)] as well as numerical simulations.

Taylor expansion of Eqs. (4) and (5) cannot be ignored. Extending the expansion even further is neither very practical nor tractable. Instead, in order to get a generalized gain function that is valid for higher intensities and into the saturation regime, it is more productive to parametrize the gain.

One widely used parametrization of the gain as a function of intensity I is

$$G(I)=\frac{G_0}{1+I/I_s}, \tag{9}$$

where G_0 is determined from the low-intensity behavior of the gain, and I_s is an empirical constant that depends on the nature of the system being investigated. It can be seen from the above equation that I_s is the value of the intensity at which the initial gain G_0 halves, and it can therefore be determined experimentally. Such a formula is often used for both conventional lasers and as well as FELs. Of course, the above formula can only be expected to give an approximate, qualitative, description of gain saturation, and a more accurate gain function $G(I)$ will depend on the details of the system being investigated.

For FELs, Dattoli *et al.* have studied this issue in some detail [8–10]. In particular, they have performed a simple calculation [8] based on an analogy with the laser rate equations for conventional lasers, to obtain the following equation for the gain of FELs:

$$G(I)=G_0\frac{(1-e^{-I/I_s})}{I/I_s}. \tag{10}$$

There are a number of assumptions made in arriving at this equation; in particular the small gain approximation is made, so that this analysis is expected to be more suited to FEL oscillators.

Equation (10) for the gain in an FEL oscillator is important because it is based on FEL dynamics. It is essentially a two-parameter parametrization for the gain, the two parameters being G_0 and I_s. The former can be determined from a SSG analysis or measurement, and the latter has to be determined by fitting to numerical simulations or experimental data. In addition, a drawback with this formula is that there is no dependence on the energy detuning α, which we know is important in determining the gain. In fact, in the limit that $I\rightarrow 0$ one cannot recover the usual SSG formula in full because the dependence on energy detuning has been removed by assuming that $\alpha=2.6$. In the determination of I_s too, it is implicit that $\alpha=2.6$ [10], whereas we have seen in the previous section that the actual value of α at saturation is likely to be close to 4.0.

Our analysis, and Eq. (7) in particular, offers the opportunity to improve on Eq. (10). For intensities at which our analysis is valid (i.e., for $\epsilon_L^2<50$) Eq. (10) should reduce to Eq. (7). Therefore, expanding Eq. (10) in a power series in I, keeping only the linear term, and comparing with Eq. (7), determines I_s analytically and also introduces α dependence into it. When we do this we get an analytic expression for I_s, which, when evaluated for the value of $\alpha=2.6$ assumed in Refs. [8–10], gives a number that is within 20% of the corresponding number using their formula for I_s [8].

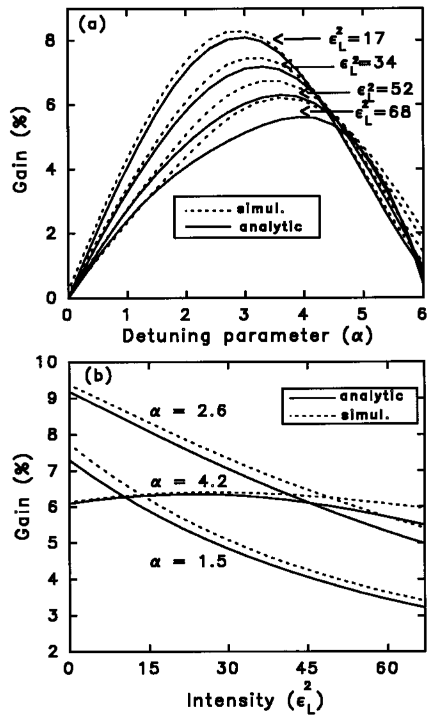

FIG. 3. Plots of the total gain (a) as a function of the detuning parameter α for different intensities ϵ_L^2 and (b) as a function of ϵ_L^2 for different α. In both cases the analytic curve is for the generalized gain function of Eq. (11). Note that in (b), for $\alpha=4.2$ the gain is nonmonotonic; it actually increases slightly before falling.

A cursory look at Eq. (7) shows that we can actually do better. We could use the ϵ_L^4 term in that equation to determine another free parameter in the gain parametrization. Note also that this is not just a refinement; the ϵ_L^4 term in Eq. (7) is needed to make the gain a nonlinear function of intensity, and, as we will show below, that is crucial in explaining an unexpected, nonmonotonic, dependence of the gain on intensity, which is seen in multiparticle simulations that do not make the small gain assumption.

We therefore propose the following three-parameter parametrization of the gain:

$$G(\alpha,\epsilon_L^2)=2j_e a_0\frac{1-\exp[-(a_2\epsilon_L^2+a_4\epsilon_L^4)]}{a_2\epsilon_L^2+a_4\epsilon_L^4}, \tag{11}$$

where the functions $a_0(\alpha)$, $a_2(\alpha)$, and $a_4(\alpha)$, determined analytically by Taylor-expanding Eq. (11) to second order in ϵ_L^2 and equating with Eq. (7), are given by

$$a_0(\alpha)=g_0(\alpha), \tag{12a}$$

$$a_2(\alpha)=-2\frac{g_2(\alpha)}{g_0(\alpha)}, \tag{12b}$$

$$a_4(\alpha)=\frac{4}{3}\left[\frac{g_2(\alpha)}{g_0(\alpha)}\right]^2-2\left[\frac{g_4(\alpha)}{g_0(\alpha)}\right]. \tag{12c}$$

Equation (11) for the *generalized gain function* is an improvement over Eq. (10) in three respects. First, it includes information from a higher-order Taylor expansion than does

Eq. (10), and is therefore expected to be more accurate; second, it includes the dependence on detuning, unlike Eq. (10), which assumes $\alpha=2.6$; third, all the parameters are determined analytically — there is no fitting to numerical data.

Figure 3(a) shows plots of $G(\alpha,\epsilon_L^2)$ as a function of α, obtained using the above parametrization as well as from the one-dimensional simulations described earlier. It can be seen that the parametrization nearly doubles the range of validity of the generalized gain function. At larger intensities there is a systematic underestimate of the actual value of the gain by around 10–15%, but the overall shape of the curve and the shifting of the peak towards higher detuning are in good agreement with the simulations.

The variation of the FEL gain $G(\alpha,\epsilon_L^2)$ as a function of the intensity ϵ_L^2 for different values of α, is shown in Fig. 3(b). Again, there is good agreement between the analytic results and numerical simulation. For different values of detuning, the gain falls at different rates. For higher values of the detuning parameter ($\alpha = 4.2$) we observe the unexpected, nonmonotonic, behavior referred to earlier in this section; the gain first *increases* slightly and then decreases. This may seem surprising, but can be understood from Fig. 1, where we see that $g_2(\alpha)$ is negative up to around $\alpha=3.7$, and beyond that it turns positive. When $g_2(\alpha)$ is positive its contribution now *adds* to that of $g_0(\alpha)$, and the net effect is an increase in the total gain. As the intensity increases further, the $g_4(\alpha)$ term, which is negative, now becomes important, so that the total gain starts decreasing. This gives rise to a nonmonotonic dependence of the gain on intensity. All the previous attempts to understand the intensity dependence of FEL gain had predicted only a monotonic fall in the gain with increasing intensity. Note that this nonmonotonic behavior is also seen in the simulations. Its origin lies in the ϵ_L^4 term in Eq. (11).

IV. BUILDUP OF INTENSITY AND SATURATION IN FEL OSCILLATORS

We now apply the generalized gain function to the study of the buildup of intensity and saturation in FEL oscillators. Typically, in an oscillator, the net gain per pass is low. The optical intensity is built up slowly, over many passes. Therefore, during any given pass the small gain approximation is good, and the analysis of the previous sections is applicable.

We therefore use the earlier analysis to model an FEL oscillator as follows. During any pass we assume that the gain is small and hence the intracavity power is nearly constant, so that Eq. (11) for the gain can be employed. With this calculation of the gain we update the value of the intracavity power, which is then used as a constant for the next pass. In this way we can model the pass-by-pass buildup of the intensity in an FEL oscillator, and compare it with one-dimensional simulations that do not make the small gain assumption.

The equation governing the buildup of intensity is given by

$$\epsilon_{L,n+1}^2=\epsilon_{L,n}^2+[G(\alpha,\epsilon_{L,n}^2)-\ell]\epsilon_{L,n}^2, \qquad (13)$$

where ℓ is the round-trip loss.

Figure 4 shows the pass-by-pass buildup of power in an

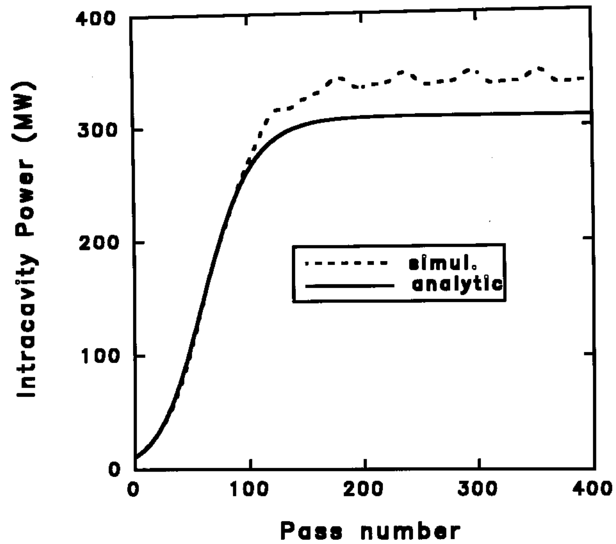

FIG. 4. Plot of the intracavity power in an FEL oscillator as a function of pass number for the analytical calculation [Eq. (13)] as well as numerical simulation.

FEL oscillator according to this simple model. Also shown is the corresponding plot from the numerical simulation. It can be seen that the agreement between the two is generally very good, especially during the growth of the intracavity power. The difference between the saturated power predicted by our analysis and the numerical simulation is typically 10%. This shows that the small gain approximation we make is a good one, and validates the application of our analysis to FEL oscillators.

The saturated intracavity intensity is an important quantity since it decides how much power we are really going to get out of the FEL. One can get the saturated intracavity intensity by numerically integrating the FEL equations. Numerical simulations, however, cannot give the insight into the functional dependencies that analytic relations can. Hence, in order to get a better understanding of saturation in FEL oscillators, it is worth attempting an analytic calculation of the saturated intracavity intensity. Our analysis makes it possible to calculate the saturated intracavity intensity $\epsilon_{L,\mathrm{sat}}^2$

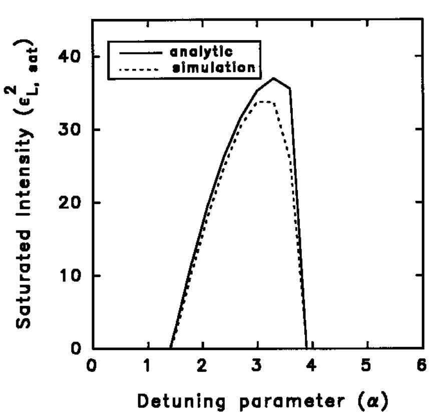

FIG. 5. Plot of the saturated intensity $\epsilon_{L,\mathrm{sat}}^2$ in an FEL oscillator as a function of the detuning parameter α. The analytic curve corresponds to the solution of Eq. (14).

as a function of the detuning parameter α by equating the gain at saturation to the loss, i.e.,

$$G(\alpha,\epsilon_{L,\mathrm{sat}}^2)=\ell. \tag{14}$$

Figure 5 shows $\epsilon_{L,\mathrm{sat}}^2$ as a function of the detuning parameter α, calculated using Eq. (14), where a round-trip loss of 7% has been assumed. As α increases from zero, initially the gain is less than the loss. Lasing starts only when the gain becomes just greater than the loss, and hence there exists a lower threshold in α below which $\epsilon_{L,\mathrm{sat}}^2=0$. As the detuning is increased $\epsilon_{L,\mathrm{sat}}^2$ initially increases, reaches a maximum, and then falls abruptly at an upper threshold beyond which the gain is again less than the loss. The corresponding numerical simulation curve is also shown in Fig. 4. We find that the agreement between the analytic calculation and the numerical simulation is quite good. The positions of the two thresholds agree well, and even the difference between the value of the maximum saturated intensity as predicted by our analysis and the numerical simulation is typically less than 10%. This suggests that our analysis could be very useful in the design of FEL oscillators, for making quick yet reliable estimates of the saturated power, and in studying the scaling of saturated intensity with total round-trip loss.

V. COMPARISON WITH EXPERIMENTS

In the previous section we have used our analysis to study the buildup and saturation of power in an FEL oscillator, and have shown that the results are in good agreement with one-dimensional simulations with typical FEL parameters. This motivates strongly the conclusion that the regime in which our analysis is valid (large signal, small gain regime) is both realistic and relevant. However, especially given the simplicity of the analysis, there may remain some questions regarding its applicability to real, operating, FELs — and hence its relevance and usefulness.

To address this issue we chose two operating FEL oscillators for which all the relevant data were readily available: the Beijing FEL and the FIREFLY FEL at Stanford. We then used Eq. (14) to predict the saturated power and energy detuning for both the FELs, and compared with the available data.

For the Beijing FEL the operating parameters were [11] $E=24$ MeV, I_{peak} = 15 A, λ_W = 3 cm, N_W = 50, $a_W=0.83$, and λ_R = 10.68 μm. The SSG and the total cavity loss were reported to be 32% and 8%, respectively. The experimental data for the optical beam radius were not reported. However, on the basis of the reported design parameters of the resonator cavity, we calculated the mean radius of the optical beam inside the undulator to be around 2 mm. To apply our analysis to this FEL we first calculated the "effective" dimensionless current density j_e from the SSG data. In this way, various gain degrading effects such as filling factor and slippage can be empirically accounted for; they are all lumped into the "effective" j_e. We then used Eq. (11) for the generalized gain function in Eq. (14) to calculate the saturated intracavity power for different α. We found that the maximum value of the saturated power was 15 MW at $\alpha=4.7$. This value of the saturated power is in good agreement with the observed value of 20 MW. There was no direct measurement of α, but the detuning can be deduced as the shift in the peak of the radiation spectrum between spontaneous and stimulated emission. This was reported to be 1.6%. For $\alpha=4.7$ this shift comes out to be 1.5%, which is in very good agreement with the experimental number.

For the FIREFLY FEL the operating parameters were [12] $E=22$ MeV, $I_{\mathrm{peak}}=14$ A, $N_W=25$, $\lambda_W=6$ cm, $a_W=1.05$, and $\lambda_R=32$ μm. The optical beam radius was 3 mm [13]. The SSG and the total cavity losses were reported [12] to be 7.0% and 4.3%, respectively. For these parameters, a similar analysis gave a maximum saturated power of 25 MW at $\alpha=4.0$. This is in good agreement with the measured saturated power of 30 MW with an experimental uncertainty of $\pm$ 30% [13]. For FIREFLY there are no data for the shift in the radiation peak, but we predict a shift of 2.5%.

So, we conclude that the predictions of our analysis are in good agreement with both the FIREFLY and Beijing FEL oscillators operating in small gain, large signal regime. This further validates the relevance and applicability of our analysis.

VI. DISCUSSION AND CONCLUSIONS

It should be emphasized that our calculation is valid in the small gain, large signal regime, which is the regime of relevance for FEL oscillators where the gain can be quite low but the intensity can, over many passes, build up to a large value. For amplifiers, where the gain is generally high, the electron and radiation dynamics cannot be decoupled and hence the present analysis is not expected to be valid.

It should also be noted that our analysis gives an expression for the generalized gain function, which is universal in nature since it is in terms of the dimensionless variables j_e and ϵ_L^2. The dependence of the gain on various FEL parameters can be extracted by simply writing the dimensionless variables in terms of FEL parameters. In this way, one can get scaling relations for the saturated intracavity power in terms of the various FEL parameters, such as wiggler length L_W, wiggler parameter a_W, wiggler period λ_W, beam energy E, etc.

There are certain gain degradation effects that we have not considered in our analysis. For example, we have not considered the energy spread in our analysis since we assume a monoenergetic electron beam. However, as shown in Ref. [10] using numerical simulations, for typical FEL oscillator parameters the gain and saturation intensity typically change by less than 10% for a relative energy spread of 1%. Similarly, the transverse emittance, which can be modeled by an equivalent longitudinal energy spread, is also not expected to change the saturation intensity significantly. To some extent gain degradation due to filling factor and slippage are empirically taken into account in our comparison with experiments by lumping them in the "effective" j_e, which is determined from the *measured* value of the SSG. Hence, the errors creeping in our analysis by ignoring these effects are expected to be typically around 10–20%.

Analytic work on the single-particle Colson equations has generally been restricted to the SSG analysis. Away from this regime analyses have usually assumed a continuous electron beam and attempted to solve the coupled Maxwell-Vlasov equations. This is in spite of the fact that numerical simulations of short wavelength FELs are based on the single-particle equations and have proven to be extremely successful. Our analysis, and the good agreement with two operating FELs, shows that for FEL oscillators one can get useful physics out of the single-particle Colson equations even away from the SSG limit.

It is true that by numerical integration of the Colson equations [Eqs. (2)], one can get more detailed information about gain and saturation in FEL oscillators. This approach, however, cannot give insight into the various functional dependencies. Our analytical calculation, on the other hand, has the advantage that it can be used to get various functional dependencies, which helps in developing a better physics insight into the system. For example, the nonmonotonic dependence of the gain on intensity, which is also seen in numerical simulations, can be qualitatively understood by simply looking at Fig. 1 obtained from our analysis (as has been discussed in Sec. III). In Fig. 3 we see that at around $\alpha=4.0$, the gain versus intensity curve is relatively flat. Thus, at higher intensity, this is the most favorable detuning for the oscillator to pick. This would perhaps explain why, in both simulations as well as the experiments reported here, for a variety of FEL parameters, the value of α at saturation is around 4.0.

It is interesting to note that our analysis can also be used to explicitly test the validity of the well known Madey gain-spread theorem [14] beyond the SSG limit. The Madey theorem relates the first moment of the electron energy change to its second moment in the wiggler in the following way:

$$\langle \Delta\mu \rangle = \frac{1}{2}\frac{d}{d\alpha}\langle \Delta\mu^2 \rangle. \tag{15}$$

Using the perturbative expansion of Eq. (4), this relation can be checked to any order in ϵ_L. We have checked that $\langle\mu_1\rangle=\langle\mu_3\rangle=\langle\mu_1\mu_2\rangle=0$. Hence, the lowest order nonvanishing term in the expression for both $\Delta\mu$ and $\Delta\mu^2$ is of second order in ϵ_L. Equating them, we get the Madey theorem in the SSG limit: $\langle\mu_2\rangle=(1/2)d/d\alpha(\langle\mu_1^2\rangle)$. However, as we go to the next-order nonvanishing term (which is of fourth order in ϵ_L), the validity of the Madey theorem demands that $\langle\mu_4\rangle=(1/2)d/d\alpha\langle\mu_2^2+2\mu_1\mu_3\rangle$. With the expressions for μ_3 and μ_4 derived by us, we find that

$$\langle\mu_4\rangle = \frac{1}{64\alpha^7}[-32\alpha^3+(4852\alpha^2)\cos\alpha+(336\alpha^2)\cos 2\alpha + (124\alpha 4\alpha^3)\sin\alpha+28\alpha\sin 2\alpha], \tag{16a}$$

$$\frac{1}{2}\frac{d}{d\alpha}\langle\mu_2^2+2\mu_1\mu_3\rangle = \frac{1}{16\alpha^7}[246+(288-37\alpha^2)\cos\alpha + (22\alpha^2-42)\cos 2\alpha + (173\alpha-3\alpha^3)\sin\alpha + (4\alpha^3-49\alpha)\sin 2\alpha]. \tag{16b}$$

Since the right-hand sides of Eqs. (16) are not identical, the Madey theorm is violated beyond the SSG limit.

In summary, we have shown that the standard single-particle perturbative analysis can be extended to give information about the detuning as well as intensity dependence of the gain. One can use this analysis to fix the parameters of the generalized gain function that is valid to larger intensities and into the saturation regime. We thus obtain a generalized gain function $G(\alpha,\epsilon_L^2)$ that is analytic, has no free parameters, is universal, and agrees well with numerical simulation, even in its prediction of an unusual nonmonotonic variation of the gain with intensity. Predictions of the saturated intensity are also in good agreement with numerical simulations as well as with experimental data from two operating FEL oscillators, the Beijing FEL and the FIREFLY FEL at Stanford. Our analysis should therefore be useful in the design of FEL oscillators.

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